

SPANNING k -ENDED TREES OF 3-REGULAR CONNECTED GRAPHS

HAMED GHASEMIAN ZOERAM¹ AND DANIEL YAQUBI²

¹Department of Pure Mathematics, Ferdowsi University of Mashhad
P. O. Box 1159, Mashhad 91775, Iran.
Email: hamed90ghasemian@gmail.com

²Department of Pure Mathematics, Ferdowsi University of Mashhad
P. O. Box 1159, Mashhad 91775, Iran.
Email: daniel_yaqubi@yahoo.es .

ABSTRACT. A vertex of degree one is called an *end-vertex* and the set of end-vertices of G is denoted by $End(G)$. For a positive integer k , a tree T be called k -ended tree if $|End(T)| \leq k$. In this paper, for each 3-regular connected graph with $|G| \geq 8$, we find a positive integer k related to order of G , such that G has a spanning k -ended tree, and with construction a sequence of 3-regular connected graphs which their orders approach to infinity with known minimum possible number of end-vertices of their spanning trees. At the end, we given a conjecture about spanning k -ended trees of 3-regular connected graphs.

1. INTRODUCTION

Throughout this article we consider only finite undirected labelled graphs without loops or multiple edges. The vertex set and edge set of graph G is denoted by $V = V(G)$ and $E = E(G)$, respectively. For $u, v \in V$ An edge joining two vertices u and v is denoted by uv or vu . The neighbourhood $N_G(v)$ or $N(v)$ of vertex v is the set of all $u \in V$ which are adjacent to v . The degree of a vertex v , denoted by $\deg_G(v) = |N_G(v)|$. The minimum degree of a graph G is denoted $\delta(G)$ and the maximum degree is denoted $\Delta(G)$. If all

2010 *Mathematics Subject Classification.* 05C05 ,05C07 .

Key words and phrases. Spanning tree, k -ended tree, branch , leaf, 3-regular graph, connected graph, triangle-free.

vertices of G have same degree k , then the graph G is called k -regular.

The distance of u and v , denoted by $d_G(u, v)$ or $d(u, v)$, is the length of a shortest path between u and v . A *Hamiltonian path* of a graph is a path passing through all vertices of the graph. A graph is *Hamiltonian-connected* if every two vertices are connected with a Hamiltonian path. In graph G , an *independent set* is a subset S of $V(G)$ such that no two vertices in S are adjacent. A maximum independent set is an independent set of largest possible size for a given graph G , This size is called the independence number of G , that denoted by $\alpha(G)$.

A vertex of degree one is called an *end-vertex*, and the set of end-vertices of G is denoted by $End(G)$. If T be tree, an end-vertex of a T is usually called a *leaf* of T and the set of leaves of T is denoted by $leaf(T)$. A spanning tree is called independence if $End(G)$ is independent in G . For a positive integer k , a tree T is said to be a k -ended tree if $|End(T)| \leq k$. We define $\sigma_k(G) = \min\{d(v_1) + \dots + d(v_k) \mid \{v_1, \dots, v_k\} \text{ is an independent set in } G\}$. Clearly, $\sigma_1(G) = \delta(G)$.

By using $\sigma_2(G)$, Ore [4] obtain the following famous theorem on Hamiltonian path. Notice that a Hamiltonian path is spanning 2-ended tree.

Theorem 1.1. [4] *Let G be a connected graph, if $\sigma_2(G) \geq |G| - 1$, then G has Hamiltonian path.*

The following theorem of Las Vergnas Broersma and Tuinstra [1] gives a similar sufficient condition for a graph G to have a spanning k -ended tree.

Theorem 1.2. [2] *Let $k \geq 2$ be an integer, and let G be a connected graph. If $\sigma_2(G) \geq |G| - k + 1$, then G has a spanning k -ended tree.*

Win [10] obtained a sufficient condition related to independent number for k -connected graph that confirms a conjecture of Las Vergnas Broersma and Tuinstra [1] gave a degree sum condition for a spanning k -ended tree.

Theorem 1.3. [10] *Let $k \geq 2$ and let G be a m -connected graph. If $\alpha(G) \leq m + k - 1$, then G has a spanning k -ended tree.*

A closure operation is useful in the study of existence of Hamiltonian cycles, Hamiltonian path and other spanning subgraphs in graph. It was first introduced by Bondy and Chavatal.

Theorem 1.4. [1] *Let G be a graph and let u and v be two nonadjacent vertices of G then,*

- (1). *Suppose $\deg_G(u) + \deg_G(v) \geq |G|$. Then G has a Hamiltonian cycle if and only if $G + uv$ has a Hamiltonian cycle.*
- (2). *Suppose $\deg_G(u) + \deg_G(v) \geq |G| - 1$. Then G has a Hamiltonian path if and only if $G + uv$ has a Hamiltonian path.*

After [1], many researchers have defined other closure concepts for various graph properties. More on k -ended tree and spanning tree can be found in [6, 7, 8, 9].

2. OUR RESULTS

Lemma 2.1. *Let T be a tree with n vertices such that $\Delta(T) \leq 3$. If $| \text{leaf}(T) | = k$ and p be the number of vertices of degree 3 in T , then $k = p + 2$.*

Proof. It is easy by the induction on p . □

Lemma 2.2. *Let G be a labelled graph and $k \geq 3$ is the smallest integer such that G has a spanning tree with k leaves like T , then no two leaves of T are adjacent in G .*

Proof. Let T be a spanning subtree of G with k leaves. Put $S = \{v_1, v_2, \dots, v_k\}$ be the set of all leaves of T . By contradiction, suppose that v_1 and v_2 are two leaves in T that adjacent in G . Consider the garph $T_1 = T + v_1v_2$, then T_1 contains a unique cycle as $C : v_1v_2c_1c_2 \dots c_\ell v_1$ where $c_i \in G$ for $1 \leq i \leq \ell$. Since $k \geq 3$ then there exist vertex $v_s \in G$ such that $v_s \notin C$. Let P be the shortest path of vertex v_s to the cycle C such that cross of the cycle C in the vertex c_j for $1 \leq j \leq \ell$.

Now, we omit the edge $c_{j-1}c_j$ of T_1 , (If $j = 1$ put $c_{j-1} = v_2$). Let $T_2 = T_1 - c_{j-1}c_j$. Then T_2 is a spanning subtree of G such that $\deg_{T_2}(c_j) = 2$. The vertices of degree one in spanning subtree T_2 is equal to the set $\{v_3, v_4, \dots, v_k\}$ either $\{v_3, v_4, \dots, v_k, c_{j-1}\}$. That is a contradict by minimality of k . □

Theorem 2.3. *Let G be a 3-regular connected graph such that $n = |G| \geq 8$. Then G is a $\lfloor \frac{2n+4}{9} \rfloor$ -ended tree.*

Proof. For positive integer k , let T be a spanning subtree of G with the k leaves such that k is minimum. If $k = 2$ then it is obvious theorem is true, so we suppose $k > 2$.

Put

$$\begin{aligned}\mathcal{A}_1 &= \{v \in V(G) \mid \deg_T(v) = 1\}; \\ \mathcal{A}_2 &= \{v \in V(G) \mid \deg_T(v) = 2\}; \\ \mathcal{A}_3 &= \{v \in V(G) \mid \deg(v) = 3\}.\end{aligned}$$

Let v be a vertex of the set \mathcal{A}_1 . Since G is a 3-regular graph, then there exist two edges as $e_1(= vv_i), e_2(= vv_j) \in E(G) - E(T)$, adjacent to v . By using of lemma 2.2, the vertices v_i and v_j belong to the set \mathcal{A}_2 . So, for each $v \in \mathcal{A}_1$ there exist two vertices of the set \mathcal{A}_2 , such that they were adjacent to v in G but not in T .

Now, choice $v \in \mathcal{A}_1$ and consider one of adjacent edges to v like vw that $w \in \mathcal{A}_2$ and $vw \in E(G) - E(T)$. The graph $T + vw$ has a cycle like $vvwv_1 \dots v_s v$ such that $v_1 \in \mathcal{A}_2$ otherwise $v_1 \in \mathcal{A}_3$ and $T + vw - wv_1$ is a spanning subtree of G with $k - 1$ leaves, it is impossible.

In other hand, no members of \mathcal{A}_1 except v (for example u) can not be adjacent to v_1 in G because, then $T_1 - wv_1$ is a spanning subtree of G with k leaves such in that v_1 and u have degree one and they are adjacent in G , it is contradiction with the lemma 2.2. We can choice the vertex w at above such that the edge $vv_1 \notin E(T)$, because if $vv_1 \in E(T)$, we choice another vertex of the set \mathcal{A}_2 like x where $vx \in E(G) - E(T)$. Now, the graph $T + vx$ contains a cycle like $vxz_1 \dots z_d v$ such that $z_1 \neq v_1$ (why). It is obviously the edge $vz_1 \notin E(T)$. So, we can suppose $vv_1 \notin E(T)$.

We have two cases:

Case I. $vv_1 \notin E(G)$, then we correspond the set $\{v_1, w, y\}$, (y is another vertex such that $y \neq w$ and $vy \in E(G) - E(T)$) to v .

Case II. If vv_1 is an edge of G that is not in T then $T + vv_1$ has a cycle like $vv_1 mu_1 \dots u_\ell v$ and again like before $m \in \mathcal{A}_2$. It is true that $m \neq w$ because if not then two cycles $vv_1 mu_1 \dots u_\ell v$ and $vvwv_1 \dots v_s v$ must cut each other, suppose that first vertex after m in cycle $vv_1 mu_1 \dots u_\ell v$ that is also in $vvwv_1 \dots v_s v$ is $v_i = u_j$ then $wv_1 \dots v_i u_{j-1} \dots u_1 w$ is a cycle in T and this is not possible.

Now as before no members of \mathcal{A}_1 , except v can not be adjacent to m in G , then we correspond $\{w, v_1, m\}$ to v .

For other vertices in the set \mathcal{A}_1 , we do corresponding like as the vertex v . Finally, we correspond for each vertices in the set \mathcal{A}_1 , one set with vertices of the set \mathcal{A}_2 , such that

each set has two elements. For example, for $v \in \mathcal{A}_1$, if consider **case I**, then w and y no ones don't appear in no correspondent set of no elements of the set \mathcal{A}_1 except vertex v ; and the third element maximum appears in one another correspondent set of another element of the set \mathcal{A}_1 .

So, we have $3 \times |\mathcal{A}_1| - \frac{1}{2} \times |\mathcal{A}_1| \leq |\mathcal{A}_2|$. If put $|\mathcal{A}_1| = k$, $|\mathcal{A}_2| = n_1$ and $|\mathcal{A}_3| = p$, by using of the lemma 2.1, we have $k = p + 2$ and since $3 \times |\mathcal{A}_1| - \frac{|\mathcal{A}_1|}{2} \leq |\mathcal{A}_2|$, this makes $\frac{5k}{2} \leq n_1$. We have

$$n = p + n_1 + k = k - 2 + n_1 + k \geq k - 2 + \frac{5k}{2} + k = \frac{9k}{2} - 2 \implies k \leq \left\lfloor \frac{2n + 4}{9} \right\rfloor.$$

□

3. SOME CONCLUDING REMARKS

In this section, we construct the sequence G_m of 3-regular graphs. For $m = 1$, consider the graph G_1 as figure 1:

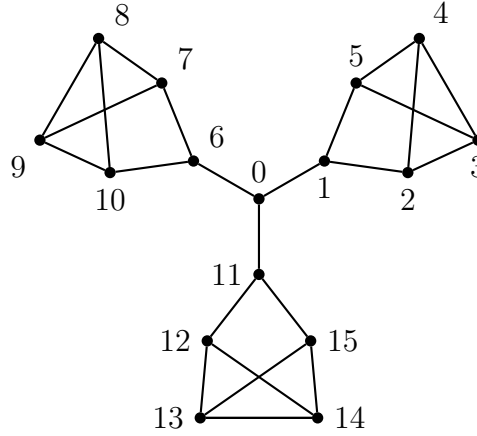


FIGURE 1. The graph G_1 with 3 branches.

Clearly G_1 has spanning subtree like T with 3 Leaves and G has no spanning subtree with less than 3 leaves. Every part of graphe G_1 , like, induce subgraph by the vertices set $\{1, 2, 3, 4, 5\}$, is called a *branch*. Then, G_1 has 3 branches. Let H be a branch of the graph G_1 with vertices $\{1, 2, 3, 4, 5\}$ and edge set $\{12, 15, 23, 24, 34, 35, 45\}$. Since the edge $\{01\}$ is a cut edge in the graph G_1 , so T must has a vertex with degree one in $V(H)$. Also, in every set-vertex of other branches of the garph G_1 , the tree T must has a vertex with degree one. So, G_1 is 3-ended tree and has no spanning tree with less than 3 leaves.

Now, we counteract 3-regular graph G_2 , consider G_1 and for each branch of it, like H

defined as before, we removed two vertices $\{3, 4\}$ and add 8 new vertices $\{v_1, \dots, v_8\}$, then we construct a new 3-regular graph with 6 branches as figure 2, such that the figure 2, is one of its three parts. Clearly $|G_2| = 16 + 3 \times 6$ and minimum number leaves in

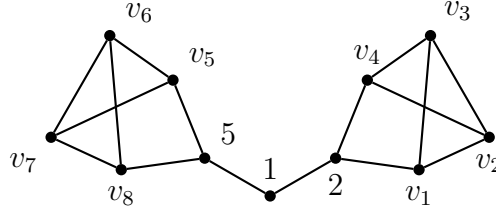


FIGURE 2. Two branch of the new graph G_2 , that construct of the graph G_1 .

every spanning subtree of G_2 is at least 2×3 and obviously G_2 has spanning subtree with 2×3 leaves.

Let the number of vertices of G_m is equal n and the number of branches of G_m is equal k , then we have the following table:

m	n	k
G_1	16	3
G_2	$16 + 3 \times 6$	2×3
G_3	$16 + 3 \times 6 + 2 \times 3 \times 6$	$2 \times 2 \times 3$
\dots	\dots	\dots
G_m	$16 + 3 \times 6 + \dots + 2^{m-2} \times 3 \times 6$	$2^{m-1} \times 3$

TABLE 1

It is obvious for each $m \in \mathbb{N}$, If the number of vertices of G_m is equal n and the number of branches of G_m is equal k , then $\frac{n+2}{6} = k$, and so G_m is $\frac{n+2}{6}$ -ended tree (such that $\frac{n+2}{6}$ is the minimum number for that G_m has spanning $\frac{n+2}{6}$ -ended tree).

Conjecture 3.1. There exist $n \in \mathbb{N}$ such that each 3-regular graph with at least n vertices has spanning $\lfloor \frac{n+2}{6} \rfloor$ -ended tree.

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